

LOCALLY n -CONNECTED COMPACTA AND UV^n -MAPS

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ABSTRACT. We provide a machinery for transferring some properties of metrizable ANR -spaces to metrizable LC^n -spaces. As a result, we show that for complete metrizable spaces the properties ALC^n , LC^n and WLC^n coincide to each other. We also provide the following spectral characterizations of ALC^n and cell-like compacta: A compactum X is ALC^n if and only if X is the limit space of a σ -complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable LC^n -spaces X_α such that all bonding projections p_α^β , as well as all limit projections p_α , are UV^n -maps. A compactum X is a cell-like (resp., UV^n) space if and only if X is the limit space of a σ -complete inverse system consisting of cell-like (resp., UV^n) metric compacta.

1. INTRODUCTION

Following [7], we say that a space M is weakly locally n -connected (briefly, WLC^n) in a space Y if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood U in M there exists a neighborhood V of x in M such that any map from \mathbb{S}^k , $k \leq n$, into V is null-homotopic in \tilde{U} , where \tilde{U} is any open in Y set with $\tilde{U} \cap M = U$ (in such a case we say that \tilde{U} is an open extension of U in Y). Dranishnikov [5] also suggested the following notion: a space M is approximately locally n -connected (briefly, ALC^n) in a space Y if $M \subset Y$ is closed and for every point $x \in M$ and its open neighborhood U in M there exists a neighborhood V of x in M such that for any open in Y extension \tilde{U} of U there exists an open in Y extension \tilde{V} of V with any map from \mathbb{S}^k , $k \leq n$, into \tilde{V} being null-homotopic in \tilde{U} . One can show that if M is metrizable (resp., compact), then M is WLC^n in a given space Y , where Y is a metrizable (resp., compact) ANR , if and only if M

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is WLC^n in any metrizable (resp., compact) ANR containing M as a closed set. The same is true for the property ALC^n . So, the definitions of WLC^n and ALC^n don't depend on the ANR -space containing M , and we say that M is WLC^n (resp., ALC^n).

Dranishnikov [7] proved that both properties WLC^n and LC^n are identical in the class of metrizable compacta. It also follows from Gutev [10] and Dugundji-Michael [9] that this remains true for complete metrizable spaces. One of the main result in this paper, Theorem 2.7, shows that all properties WLC^n , LC^n and ALC^n coincide for completely metrizable spaces. The proof of Theorem 2.7 is based on the technique, developed in Section 2, for transferring properties of metrizable ANR 's to LC^n -subspaces (in this way well known properties of metrizable LC^n -spaces can be obtained from the corresponding properties of metrizable ANR 's, see for example Proposition 2.2). Section 2 contains also a characterization of metrizable LC^n -spaces whose analogue for ANR 's was established by Nhu [12].

It is well known that the class of metrizable LC^n -spaces are exactly absolute neighborhood extensors for $(n + 1)$ -dimensional paracompact spaces, and this is not valid for non-metrizable spaces. Outside the class of metrizable spaces we have the following characterization of ALC^n compacta (Theorem 3.1): A compactum X is ALC^n if and only if X is the limit space of a σ -complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable LC^n -spaces X_α such that all bonding projections p_α^β , as well all limit projections p_α , are UV^n -maps. A similar spectral characterization is obtained for cell-like or UV^n compacta, see Theorem 3.3. Both Theorem 3.1 and Theorem 3.3 provide different classes of compacta \mathcal{C} and corresponding classes of maps \mathcal{M} adequate with \mathcal{C} in the following sense (see Shchepin [18]): A compactum X belongs to \mathcal{C} if and only if X is the limit space of a σ -complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable $X_\alpha \in \mathcal{C}$ with all bonding projections p_α^β being from \mathcal{M} . For example, according to Theorem 3.1, the class of ALC^n compacta is adequate with the class of UV^n -maps.

Recall that a closed subset $A \subset X$ is UV^n in X (resp., cell-like in X) if every neighborhood U of A in X contains a neighborhood V of A such that, for each $0 \leq k \leq n$, any map $f: \mathbb{S}^k \rightarrow V$ is null-homotopic in U (resp., A is contractible in every neighborhood of A in X). A space X is said to be UV^n (resp., cell-like) provided it is UV^n (resp., cell-like) in some ANR -space containing X as a closed set. It is well known that for metrizable or compact X this definition does not depend on the ANR -spaces containing X as a closed set, see for example [17]. A map

$f: X \rightarrow Y$ between compact spaces is called UV^n (resp., cell-like) if all fibres of f are UV^n (resp., cell-like).

Theorem 3.1 yields that any compact LC^n -space is ALC^n . It is interesting to find an example of an ALC^n -compactum which is not LC^n (obviously, such a compactum should be non-metrizable).

2. METRIZABLE ALC^n -SPACES

We are going to establish some properties of metric LC^n -spaces using the corresponding properties of metric ANR -spaces. Recall that a map $f: X \rightarrow Y$ is n -invertible if for every Tychonoff space Z with $\dim Z \leq n$ and any map $g: Z \rightarrow Y$ there is a map $h: Z \rightarrow X$ such that $g = f \circ h$.

The next theorem follows from a stronger result due to Pasynkov [15, Theorem 6], and its proof is based on Dranishnikov results [6, Theorem 1] and [7, Theorem 1.2] (such a result concerning the extension dimension with respect to quasi-finite complexes was established in [14, Proposition 2.7]). We provide here a proof based on factorization theorems.

Theorem 2.1. *Every Tychonoff space M is the image of a Tychonoff space X with $\dim X \leq n$ under a perfect n -invertible map. In case M is metrizable, X can be supposed to be also a metrizable space with $w(X) = w(M)$.*

Proof. Let M be a Tychonoff space of weight τ . Consider all couples (Z_α, f_α) , where Z_α is a Tychonoff space of weight $w(Z_\alpha) \leq w(\beta M)$, $\dim Z_\alpha \leq n$ and f_α is a map from Z_α into M (here βM is the Čech-Stone compactification of M). Denote by Z the disjoint sum of all spaces Z_α . Obviously, there is a natural map $f: Z \rightarrow M$ such that $f|_{Z_\alpha} = f_\alpha$ for all α . Let $\tilde{f}: \beta Z \rightarrow \beta M$ be the extension of f . Then, by the Mardešić's factorization theorem [11], there exists a compactum \tilde{X} of weight $w(\tilde{X}) \leq w(\beta M)$ with $\dim \tilde{X} \leq \dim \beta Z = n$ and maps $h: \beta Z \rightarrow \tilde{X}$, $\tilde{g}: \tilde{X} \rightarrow \beta M$ such that $\tilde{g} \circ h = \tilde{f}$. Let $X = \tilde{g}^{-1}(M)$ and $g = \tilde{g}|_X$. According to Corollary 6 and Main Theorem from [15], $\dim X \leq n$. To show that g is n -invertible, suppose $f_0: Z_0 \rightarrow M$ is a map with $\dim Z_0 \leq n$. Applying again the Mardešić's factorization theorem for the map $\tilde{f}_0: \beta Z_0 \rightarrow \beta M$, we obtain a compactum K and maps $h_1: \beta Z_0 \rightarrow K$ and $f_2: K \rightarrow \beta M$ such that $\dim K \leq n$, $w(K) \leq w(\beta M)$ and $f_2 \circ h_1 = \tilde{f}_0$. Then, as above, $Z' = f_2^{-1}(M)$ is a space of dimension $\leq n$ and weight $\leq w(\beta M)$. So, there exists α^* and a homeomorphism $j: Z' \rightarrow Z$ such that $j(Z') = Z_{\alpha^*}$ and $f_{\alpha^*} \circ j = f_2|_{Z'}$. Consequently, $h \circ j$ is a map from Z' to X with $f_2|_{Z'} = g \circ h \circ j$. Finally, $h \circ j \circ h_1$ is a map from Z_0 to X such that $g \circ h \circ j \circ h_1 = f_0$.

If M is metrizable, the proof is simpler. Indeed, in this case $P = \tilde{f}^{-1}(M)$ is a space of dimension $\leq n$ and the restriction $\tilde{f}|_P$ is a perfect map. So, by Pasynkov's factorization theorem [16], there exists a metrizable space X and maps $h: P \rightarrow X$, $g: X \rightarrow M$ such that $g \circ h = \tilde{f}$, $w(X) \leq w(M)$ and $\dim X \leq n$. Then g is a perfect map because so is $\tilde{f}|_P$, and according to the above arguments, g is n -invertible. \square

The next proposition shows that Theorem 2.1 allows some properties of metrizable ANR -spaces to be transferred to metrizable LC^n -spaces.

Proposition 2.2. *Let M be a metrizable LC^n -space and α an open cover of M . Then there exists an open cover β of M refining α such that for any two β -near maps $f, g: Z \rightarrow M$ defined on a metrizable space Z of dimension $\leq n$ any β -homotopy $H: A \times [0, 1] \rightarrow M$ between $f|_A$ and $g|_A$, where A is closed in Z , can be extended to an α -homotopy $\tilde{H}: Z \times [0, 1] \rightarrow M$ connecting f and g .*

Proof. We embed M as a closed subset of a metrizable ANR -space P and let $p: Y_P \rightarrow P$ be a perfect $(n+1)$ -invertible surjection such that Y_P is a metrizable space of dimension $\leq n+1$ (see Theorem 2.1). Since M is LC^n , there is an open set G in Y_P containing $p^{-1}(M)$ and a map $q: G \rightarrow M$ extending the restriction $p|_{p^{-1}(M)}$. Then there exists an open set $W \subset P$ containing M with $p^{-1}(W) \subset G$ (recall that p is a perfect map). Obviously W is also an ANR -space containing M as a closed set. So, without losing generality, we may assume that $W = P$, $G = Y_P$ and q is a map from Y_P onto M . Now, for every open $U \subsetneq M$, let $\tilde{U} = P \setminus (p(q^{-1}(M \setminus U)))$. The set \tilde{U} is non-empty and open in P , $\tilde{U} \cap M = U$ and $p^{-1}(\tilde{U}) \subset q^{-1}(U)$.

If α is an open cover of M consisting of proper subsets of M , the set $L = \bigcup \{\tilde{U} : U \in \alpha\}$ is open in P and contains M . So, L is also an ANR and $\tilde{\alpha} = \{\tilde{U} : U \in \alpha\}$ is an open cover of L . According to the properties of metrizable ANR 's (see for example [13, chapter IV, Theorem 1.2]), there exists an open cover $\tilde{\beta}$ of L with the following property: for any two β -near maps $h_1, h_2: Z \rightarrow M$ defined on a metrizable space Z any β -homotopy $H: A \times [0, 1] \rightarrow M$ between $h_1|_A$ and $h_2|_A$, where A is closed in Z , can be extended to an α -homotopy $F: Z \times [0, 1] \rightarrow M$ connecting h_1 and h_2 . Then $\beta = \{V \cap M : V \in \tilde{\beta}\}$ is an open cover of M refining α and has the desired property. Indeed, suppose $f, g: Z \rightarrow M$ are two β -near maps and $H: A \times [0, 1] \rightarrow M$ is a β -homotopy between $f|_A$ and $g|_A$, where Z is a metrizable space of dimension $\leq n$ and A is closed in Z . According to the choice of $\tilde{\beta}$, H can be extended to an $\tilde{\alpha}$ -homotopy

$F: Z \times [0, 1] \rightarrow L$ connecting f and g . Since $\dim Z \times [0, 1] \leq n + 1$ and p is $(n + 1)$ -invertible, there exists a lifting $F_1: Z \times [0, 1] \rightarrow p^{-1}(L)$ of F . Finally, $\tilde{H} = q \circ F_1$ is an α -homotopy between f and g . \square

We also need the following property of metrizable LC^n -spaces.

Proposition 2.3. *Suppose both M and P are metrizable LC^n -spaces with $M \subset P$ being closed. Then there exists an open set $U \subset P$ containing M with the following property: If Z is a metrizable space with $\dim Z \leq n$ and $h: Z \rightarrow U$ is a map, there exists a map $h_1: Z \rightarrow M$ such that h and h_1 are homotopic in P .*

Proof. Let $p: Y_P \rightarrow P$ be a perfect $(n + 1)$ -invertible surjection such that Y_P is a metrizable space of dimension $\leq n + 1$, and $q: p^{-1}(W) \rightarrow M$ extends the restriction $p|_{p^{-1}(W)}$, where $W \subset P$ is an open set containing M (see the proof of Proposition 2.2). Since W (as an open subset of P) is LC^n , we can apply Proposition 2.2 (with $\alpha = \{W\}$ and A a point), to obtain that W has an open cover β such that any two β -near maps from Z into W are homotopic. For every $V \in \beta$ let $G_V = V \setminus p(q^{-1}(M \setminus (M \cap V)))$ and $U = \bigcup \{G_V : V \in \beta\}$. Because β covers M , $M \subset U$. If $h: Z \rightarrow U$ is any map, let $h_1: Z \rightarrow M$ be the map $h_1 = q \circ \tilde{h}$, where $\tilde{h}: Z \rightarrow p^{-1}(U)$ is a lifting of h . It is easily seen that the maps h and h_1 are β -near. So, h and h_1 are homotopic in W , and hence in P . \square

Corollary 2.4. *If M and P are metrizable LC^n -spaces such that $M \subset P$ is closed, then M is ALC^n in P .*

Proof. Since M is LC^n , for every $x \in M$ and its open neighborhood U in M there exists a neighborhood V of x in M such that any map of a k -sphere, $k \leq n$, into V is contractible in U . Let $\tilde{U} \subset P$ and $G \subset \tilde{U}$ be open in P extensions of U and V , respectively. Then both V and G are LC^n (as open subsets of LC^n -spaces), and V is closed in G . So, there is an open extension $\tilde{V} \subset G$ satisfying the conclusion from Proposition 2.4. Consequently, any map $g: \mathbb{S}^k \rightarrow \tilde{V}$, $0 \leq k \leq n$, is homotopic in G to a map $g_1: \mathbb{S}^k \rightarrow V$. Since g_1 is homotopic in U to a constant map, we obtain that g is homotopic in \tilde{U} to a constant map. \square

Proposition 2.5. *Let P be metrizable and $M \subset P$ be a closed and LC^n -set. Then every closed set $A \subset M$ is UV^n in M provided A is UV^n in P .*

Proof. Let $p: Y_P \rightarrow P$ be a perfect $(n + 1)$ -invertible surjection such that Y_P is a metrizable space of dimension $\leq n + 1$, and $q: p^{-1}(W) \rightarrow M$ extends the restriction $p|_{p^{-1}(W)}$, where $W \subset P$ is an open set

containing M . Suppose $U \subset M$ is an open set containing A and let $\tilde{U} = W \setminus p(q^{-1}(M \setminus U))$. Obviously, $\tilde{U} \subset W$ is an open extension of U . Since $A \in UV^n(P)$, there is an open set $\tilde{V} \subset \tilde{U}$ containing A such that any map from \mathbb{S}^k with $0 \leq k \leq n$ to \tilde{V} can be extended to a map from \mathbb{B}^{k+1} to \tilde{U} . Let $V = M \cap \tilde{V}$ and $g: \mathbb{S}^k \rightarrow V$ be a map. Then extend g to a map $g_1: \mathbb{B}^{k+1} \rightarrow \tilde{U}$ and take a lifting $g_2: \mathbb{B}^{k+1} \rightarrow p^{-1}(\tilde{U})$ of g_1 . Finally, $\tilde{g} = q \circ g_2$ is a map from \mathbb{B}^{k+1} to U extending g . \square

Next lemma is a non-compact analogue of Lemma 2.1 from [7].

Lemma 2.6. *Suppose both M and P are metric LC^{n-1} -spaces such that M is a closed subset of P . Then for every $\epsilon > 0$ there exists a neighborhood $U_\epsilon(M)$ in P such that for any map $\varphi: (Q, Q_0) \rightarrow (U_\epsilon(M), M)$, where (Q, Q_0) is a polyhedral pair, there exists a map $\psi: (Q^{(n)}, Q_0) \rightarrow M$ such that $\psi|_{Q_0} = \varphi|_{Q_0}$ and both φ and ψ are ϵ -close.*

Proof. The proof is similar to that one of Proposition 2.5, the only difference is that the map $p: Y_P \rightarrow P$ is n -invertible and $\dim Y_P \leq n$. We take an open cover ω of M with each $V \in \omega$ having a diameter $< \epsilon$. For every $V \in \omega$ consider the set $\tilde{V} = P \setminus p(q^{-1}(M \setminus V))$ and let $U_\epsilon(M) = \bigcup \{\tilde{V} : V \in \omega\}$. If $\varphi: (Q, Q_0) \rightarrow (U_\epsilon(M), M)$, we first lift $\varphi|_{Q^{(n)}}$ to a map $\varphi_1: Q^{(n)} \rightarrow Y_P$ and define $\psi: (Q^{(n)}, Q_0) \rightarrow M$ to be the map with $\psi|_{Q_0} = \varphi|_{Q_0}$ and $\psi|_{Q^{(n)}} = q \circ \varphi_1$. Obviously, ψ satisfies the required conditions. \square

Now, we are in a position to prove the main theorem in this section.

Theorem 2.7. *For a complete metric space (M, d) the following are equivalent:*

- (i) M is LC^n ;
- (ii) M is ALC^n ;
- (iii) M is WLC^n .

Proof. Implication (i) \Rightarrow (ii) follows from Corollary 2.4, and implication (ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). We are going to prove this implication by induction. Since M is LC^{-1} (there is no such thing as a (-1) -sphere), we can suppose that M is LC^{n-1} and WLC^n . We embed M as a closed subset of a complete metric ANR -space P and consider the following relation between the open subsets of M : $V \alpha U$ if $V \subset U$ and every map from \mathbb{S}^k , $k \leq n$, into V is null-homotopic in \tilde{U} , where \tilde{U} is any open extension of U in P . It follows from [9, Theorem 1] the existence

a complete metric ρ on M generating its topology such that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ with $B_{\delta(\epsilon)}^\rho(x) \subset B_\epsilon^\rho(x)$ for all $x \in M$, where $B_\epsilon^\rho(x) = \{y \in M : \rho(x, y) < \epsilon\}$. The metric ρ can be extended to a complete metric $\tilde{\rho}$ on P , see [1]. We fix a sequence $\{G_k\}_{k \geq 1}$ of open subsets of P containing M with $\bigcap_{k=1}^\infty G_k = M$, and define by induction a decreasing sequence $\{W_k\}_{k \geq 1}$ of open subsets of P such that $W_k \subset U_{2^{-k}}(M) \cap G_k$, where $U_{2^{-k}}(M)$ is a neighborhood of M in P corresponding to 2^{-k} (see Lemma 2.6). Let $\{g_k\}_{k \geq 1}$ be a sequence of continuous functions $g_k : P \rightarrow [0, 1]$ such that $g_k(M) = 1$ and $g_k(P \setminus W_k) = 0$.

Then the equality $\varrho(x, y) = \tilde{\rho}(x, y) + \sum_{k=1}^\infty \frac{|g_k(x) - g_k(y)|}{2^k}$ provides a metric on P . It easily seen that ϱ is a complete metric generating the topology of P . Moreover, $\varrho(x, y) = \tilde{\rho}(x, y) = \rho(x, y)$ for all $x, y \in M$. For any $x \in M$ and $y \in P \setminus W_k$ we have

$$\varrho(x, y) = \tilde{\rho}(x, y) + \sum_{j=1}^{j=k-1} \frac{|1 - g_j(y)|}{2^j} + \sum_{j=k}^\infty \frac{1}{2^j} \geq \frac{1}{2^{k-1}}.$$

Hence, $\varrho(M, P \setminus W_k) \geq 2^{-k+1}$ for all k . For every $\epsilon > 0$ let $k_\epsilon = \min\{k : 2^{-k} < \epsilon\}$, $\eta(\epsilon) = 2^{-k_\epsilon+1}$ and $B_\epsilon^\varrho(M) = \{y \in P : \varrho(y, M) < \epsilon\}$.

Claim 1. Let (Q, Q_0) be a polyhedral pair and $\epsilon > 0$. Then for any map $\varphi : Q \rightarrow B_{\eta(\epsilon)}^\varrho(M)$ with $\varphi(Q_0) \subset M$ there exists a map $\psi : Q^{(n)} \cup Q_0 \rightarrow M$ such that $\psi|_{Q_0} = \varphi|_{Q_0}$ and ψ is ϵ -close to φ .

Since $\varrho(M, P \setminus W_{k_\epsilon}) \geq 2^{-k_\epsilon+1} = \eta(\epsilon)$, $B_{\eta(\epsilon)}^\varrho(M) \subset W_{k_\epsilon}$. Hence, $B_{\eta(\epsilon)}^\varrho(M) \subset U_{2^{-k_\epsilon}}(M)$, and according to Lemma 2.6, there exists a map such that $\psi : Q^{(n)} \cup Q_0 \rightarrow M$ such that $\psi|_{Q_0} = \varphi|_{Q_0}$ and ψ is (2^{-k_ϵ}) -close to φ . Then the inequality $2^{-k_\epsilon} < \epsilon$ completes the proof of Claim 1.

To prove that M is LC^n , we introduce the following notation: If $U \subset M$ is open and $\gamma > 0$, then $E_\gamma(U)$ denotes any open extension of U in P which is contained in the set $B_\gamma^\varrho(M)$. Since M is WLC^n , according to the choice of the function δ , any map $\varphi : \mathbb{S}^n \rightarrow M$ with $\text{diam} \varphi(\mathbb{S}^n) < \delta(\epsilon)$ can be extended to a map $\tilde{\varphi} : \mathbb{B}^{n+1} \rightarrow E_\gamma(B_\epsilon^\rho(\varphi(\mathbb{S}^n)))$, where $\gamma > 0$ is arbitrary. Now, we proceed as in the proof of Lemma 2.5 from [7]. Fix $\epsilon > 0$ and a map $f : \mathbb{S}^n \rightarrow M$ with $\text{diam} f(\mathbb{S}^n) < \delta(\epsilon/2)$. We are going to show that f can be extended to a map $\tilde{f} : \mathbb{B}^{n+1} \rightarrow M$ such that $\text{diam} \tilde{f}(\mathbb{B}^{n+1}) \leq 10\epsilon$. The map \tilde{f} will be obtained as limit of a sequence $\{f_k : \mathbb{B}^{n+1} \rightarrow P\}$, and this sequence will be constructed by induction together with a sequence of triangulations $\{\tau_k\}$ of \mathbb{B}^{n+1} such that $f_k(\tau_k^{(n)}) \subset M$ for all k . To start the induction, we choose τ_1 to be

\mathbb{B}^{n+1} , considered as one $(n+1)$ -dimensional simplex, and let $f_1: \mathbb{B}^{n+1} \rightarrow E_{\eta(\delta(2^{-1}\epsilon)/3)}(B_\epsilon^g(z))$ be a map extending f , where $z \in f(\mathbb{S}^n)$. Suppose a triangulation τ_k of \mathbb{B}^{n+1} and a map $f_k: \mathbb{B}^{n+1} \rightarrow B_{\eta(\delta(2^{-k}\epsilon)/3)}^g(M)$ are already constructed with $f_k(\tau_k^{(n)}) \subset M$. Denote by τ_{k+1} any subdivision of τ_k such that $\text{diam} f_k(\sigma) < \delta(2^{-k}\epsilon)/3$ for any simplex $\sigma \in \tau_{k+1}$. According to Claim 1, there exists a map $g_k: \tau_{k+1}^{(n)} \rightarrow M$ such that g_k is $(\delta(2^{-k}\epsilon)/3)$ -close to the restriction $f_k|_{\tau_{k+1}^{(n)}}$ and $g_k|_{\tau_k^{(n)}} = f_k|_{\tau_k^{(n)}}$. Therefore, $\text{diam} g_k(\partial\sigma) < \delta(2^{-k}\epsilon)$ for any $\sigma \in \tau_{k+1}$. Hence, $g_k|_{\partial\sigma}$ can be extended to a map $g_k^\sigma: \sigma \rightarrow E_{\eta(\delta(2^{-k-1}\epsilon)/3)}(B_{\epsilon/2^k}^\rho(g_k(\partial\sigma)))$, $\sigma \in \tau_{k+1}$. Finally, we define $f_{k+1}: \mathbb{B}^{n+1} \rightarrow B_{\eta(\delta(2^{-k-1}\epsilon)/3)}^g(M)$, $f_{k+1}|_\sigma = g_k^\sigma$. Because $\delta(\epsilon) \leq \epsilon$ and $\eta(\epsilon) \leq \epsilon$, for every $\sigma \in \tau_{k+1}$ we have $\text{diam} f_{k+1}(\sigma) \leq \text{diam} g_k(\partial\sigma) + \epsilon/2^k + \eta(\delta(2^{-k-1}\epsilon)/3) \leq \delta(2^{-k}\epsilon) + \epsilon/2^k + \eta(\delta(2^{-k-1}\epsilon)/3) \leq 3\epsilon/2^k$. Now, for every $x \in \mathbb{B}^{n+1}$ take $\sigma \in \tau_{k+1}$ containing x and $y \in \partial\sigma$. Then $\varrho(f_k(x), f_{k+1}(x)) \leq \varrho(f_{k+1}(x), f_{k+1}(y)) + \varrho(f_k(y), f_{k+1}(y)) + \varrho(f_k(y), f_k(x))$. Consequently, $\varrho(f_k(x), f_{k+1}(x)) \leq 3\epsilon/2^k + 2\delta(2^{-k}\epsilon)/3 \leq 4\epsilon/2^k$ for all $x \in \mathbb{B}^{n+1}$. The last inequality implies that the sequence $\{f_k(x)\}$ is convergent for all $x \in M$. So, the limit map $\tilde{f}: \mathbb{B}^{n+1} \rightarrow P$ is well defined and $\varrho(f_1(x), \tilde{f}(x)) \leq 4\epsilon$. Then for every $x, y \in \mathbb{B}^{n+1}$ we have $\varrho(\tilde{f}(x), \tilde{f}(y)) \leq \varrho(\tilde{f}(x), f_1(x)) + \varrho(f_1(x), f_1(y)) + \varrho(f_1(y), \tilde{f}(y)) \leq 4\epsilon + \epsilon + \eta(\delta(\epsilon/2)/3) + 4\epsilon \leq 10\epsilon$. Since $f_k(\mathbb{B}^{n+1}) \subset B_{\eta(\delta(2^{-k}\epsilon)/3)}^g(M)$ and $\lim \eta(\delta(2^{-k}\epsilon)/3) = 0$, $\tilde{f}(\mathbb{B}^{n+1}) \subset M$. This completes the proof. \square

A sequence of open covers $\mathcal{U} = (\mathcal{U}_k)_{k \in \mathbb{N}}$ of a metric space (M, d) is called a *zero-sequence* if $\lim_{k \rightarrow \infty} \text{mesh} \mathcal{U}_k = 0$. For any such a sequence we define $\text{Tel}(\mathcal{U}) = \bigcup_{k \in \mathbb{N}} N(\mathcal{U}_k \cup \mathcal{U}_{k+1})$. Here $N(\mathcal{U}_k \cup \mathcal{U}_{k+1})$ is the nerve of $\mathcal{U}_k \cup \mathcal{U}_{k+1}$ with \mathcal{U}_k and \mathcal{U}_{k+1} considered as disjoint sets. For any $\sigma \in \text{Tel}(\mathcal{U})$ let $s(\sigma) = \max\{s : \sigma \in N(\mathcal{U}_s \cup \mathcal{U}_{s+1})\}$.

We complete this section by a characterization of metrizable LC^n -spaces similar to the characterization of metrizable ANR -spaces provided in [12] (see also [17, Theorem 6.8.1]).

Proposition 2.8. *A metric space (M, d) is LC^n if and only if it has a zero-sequence \mathcal{U} of open covers such that any map $f_0: K^{(0)} \rightarrow M$ with $f(U) \in U$, $U \in K^{(0)}$, where K is a subcomplex of $\text{Tel}(\mathcal{U})$, extends to a map $f: K^{(n+1)} \rightarrow M$ satisfying the following condition:*

- (*) *For any sequence $\{\sigma_k\}$ of simplexes of $K^{(n+1)}$ with $s(\sigma_k) \rightarrow \infty$ we have $\lim_{k \rightarrow \infty} \text{diam}(f(\sigma_k)) = 0$.*

Proof. Suppose M is LC^n and embed (M, d) isometrically in a metric ANR -space (P, ρ) as a closed subset. According to the proof of

Theorem 2.1, there are metrizable space Y_P and two maps $p: Y_P \rightarrow P$ and $q: Y_P \rightarrow M$ such that $\dim Y_P \leq n + 1$, p is $(n + 1)$ -invertible and q extends the map $p|_{p^{-1}(M)}$. Using the proof of Nhu's theorem [12, Theorem 1.1] for ANR 's, we can find a zero-sequence $\mathcal{V} = (\mathcal{V}_k)_{k \in \mathbb{N}}$ of P such that any map $h_0: K^{(0)} \rightarrow P$ with $h_0(V) \in V$ for each $V \in K^{(0)}$, where K is a subcomplex of $\text{Tel}(\mathcal{U})$, extends to a map $h: |K| \rightarrow P$ such that $\lim_{k \rightarrow \infty} \text{diam}(h(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplex of K with $s(\sigma_k) \rightarrow \infty$. Let us show that the sequence $(\mathcal{U}_k)_{k \in \mathbb{N}}$, $\mathcal{U}_k = \{V \cap M : V \in \mathcal{V}_k\}$, is as required. Indeed, for each $U = V \cap M \in \mathcal{U}_k$ define $W(U) = V \setminus (p(q^{-1}(M \setminus U)))$ and consider the open families $\mathcal{W}_k = \{W(U) : U \in \mathcal{U}_k\}$, $k \in \mathbb{N}$. We may assume that each U is a proper subset of M , so $W_k \neq \emptyset$ for all k . Note that \mathcal{W}_k may not cover P , but any \mathcal{W}_k covers M . Moreover, $\text{mesh} \mathcal{W}_k \leq \text{mesh} \mathcal{V}_k$, so $\lim_{k \rightarrow \infty} \text{mesh} \mathcal{W}_k = 0$. If K is a subcomplex of $\text{Tel}(\mathcal{U})$, take any map $f_0: K^{(0)} \rightarrow M$ with $f_0(U) \in U$. Hence, f_0 extends to a map $g: |K| \rightarrow P$ such that $\lim_{k \rightarrow \infty} \text{diam}(g(\sigma_k)) = 0$ for any sequence $\{\sigma_k\}$ of simplexes of K with $s(\sigma_k) \rightarrow \infty$. Finally, let $f: |K^{(n+1)}| \rightarrow M$ be the map $q \circ \tilde{g}$, where $\tilde{g}: |K^{(n+1)}| \rightarrow Y_P$ is a lifting of g . To show that f satisfies condition $(*)$, fix an $\epsilon > 0$, a cover \mathcal{W}_m with $\text{mesh}(\mathcal{W}_m) < \epsilon$ and a sequence $\{\sigma_k\} \subset K^{(n+1)}$ with $\lim_{k \rightarrow \infty} s(\sigma_k) = \infty$. Then $\lim_{k \rightarrow \infty} \text{mesh}(g(\sigma_k)) = 0$, so there exists k_0 such that $g(\sigma_k) \subset \bigcup \mathcal{W}_m$ for all $k \geq k_0$ (recall that $g(U) \in M$ for all $U \in K^{(0)}$). Hence, for any two different points $x, y \in \sigma_k$ with $k \geq k_0$ we have $g(x) \in W(U_x)$ and $g(y) \in W(U_y)$ for some $W(U_x), W(U_y) \in \mathcal{W}_m$. So, according to the definition of $W(U)$, $f(x) \in U_x$ and $f(y) \in U_y$. Therefore, $\rho(f(x), g(x)) \leq \text{diam}(W(U_x)) < \epsilon$ and, similarly, $\rho(f(y), g(y)) < \epsilon$. Consequently, $d(f(x), f(y)) < 2 \cdot \epsilon + \text{diam}(g(\sigma_k))$.

To prove the other implication, embed M as a closed subset of a metrizable space Z with $\dim Z \setminus M \leq n + 1$ and follow the proof of implication $(iii) \Rightarrow (i)$ from [12, Theorem 1.1] to obtain that M is a retract of a neighborhood W_1 of M in Z (the only difference is that in Fact 1.2 from [12] we take the cover \mathcal{V} of $W_1 \setminus M$ to be of order $\leq n + 1$, so the nerve $N(\mathcal{V})$ is a complex of dimension $\leq n + 1$). Then by [13, chapter V, Theorem 3.1], M is LC^n . \square

3. UV^n -MAPS AND ALC^n -SPACES

In this section we provide spectral characterizations of non-metrizable ALC^n -compacta and cell-like compacta. Recall that a map $f: X \rightarrow Y$ between compact spaces is said to be soft [18] if for every compactum Z , its closed subset $A \subset Z$ and maps $h: A \rightarrow X$ and $g: Z \rightarrow Y$ with $f \circ h = g|_A$ there exists a lifting $\bar{g}: Z \rightarrow X$ of g extending h .

Theorem 3.1. *A compactum X is ALC^n if and only if X is the limit space of a σ -complete inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compact metrizable LC^n -spaces X_α such that all bonding projections p_α^β , as well as all limit projections p_α , are UV^n -maps.*

Proof. Suppose that X is the limit space of an inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ such that each X_α is a metric LC^n -compactum and all p_α^β are UV^n -maps. We embed X in a Tychonoff cube \mathbb{I}^B , where $\mathbb{I} = [0, 1]$ and the cardinality of B is equal to τ . According to Shchepin's spectral theorem [18], we can assume that B is the union of countable sets B_α , $\alpha \in A$, such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$, $B_\gamma = \bigcup \{B_{\gamma(k)} : k = 1, 2, \dots\}$ for any chain $\gamma(1) < \gamma(2) < \dots$ with $\gamma = \sup\{\gamma(k) : k \geq 1\}$, and each $p_\alpha^\beta : X_\beta \rightarrow X_\alpha$ is the restriction of the projection $q_\alpha^\beta : \mathbb{I}^{B_\beta} \rightarrow \mathbb{I}^{B_\alpha}$. So, each $X_\alpha = q_\alpha(X)$ is a subset of \mathbb{I}^{B_α} , where q_α denotes the projection $q_\alpha : \mathbb{I}^B \rightarrow \mathbb{I}^{B_\alpha}$. We also denote $q_\alpha|_X$ by p_α . Choose $x_0 \in X$ and its neighborhood $U \subset X$. There exists $\alpha_0 < \tau$ and an open set $U_0 \subset X_{\alpha_0}$ with $x_0 \in p_{\alpha_0}^{-1}(U_0) \subset \overline{p_{\alpha_0}^{-1}(U_0)} \subset U$. Since X_{α_0} is LC^n , there exists a neighborhood V_0 of $x_{\alpha_0} = p_{\alpha_0}(x_0)$ such that for all $k \leq n$ the inclusion $j_0 : V_0 \hookrightarrow U_0$ generates trivial homomorphisms $j_0^* : \pi_k(V_0) \rightarrow \pi_k(U_0)$ between the homotopy groups. Let $V = p_{\alpha_0}^{-1}(V_0)$ and \tilde{U} be an open set in \mathbb{I}^B extending U . Choose a finite family $\omega = \{W_1, \dots, W_m\}$ of open sets from the ordinary base of \mathbb{I}^B such that $W = \bigcup_{i=1}^m W_i$ covers $\overline{p_{\alpha_0}^{-1}(U_0)}$ and $W \subset \tilde{U}$. Then we can find $\alpha_1 > \alpha_0$ with $q_{\alpha_1}^{-1}(q_{\alpha_1}(W_i)) = W_i$, $i \leq m$. Denote $V_1 = (p_{\alpha_0}^{\alpha_1})^{-1}(V_0)$ and $U_1 = (p_{\alpha_0}^{\alpha_1})^{-1}(U_0)$. Obviously, $q_{\alpha_1}(W)$ is open in $\mathbb{I}^{B_{\alpha_1}}$ containing U_1 . Take an open in $\mathbb{I}^{B_{\alpha_1}}$ extension \tilde{V}_1 of V_1 with $\tilde{V}_1 \subset q_{\alpha_1}(W)$. Since \tilde{V}_1 is an ANR and V_1 is LC^n (as an open subset of X_{α_1}), by Proposition 2.3, there exists an open in $\mathbb{I}^{B_{\alpha_1}}$ extension G of V_1 which is contained in \tilde{V}_1 with the following property: for every map $h : Z \rightarrow G$, where Z is at most n -dimensional metric space, there exists a map $h_1 : Z \rightarrow V_1$ such that h and h_1 are homotopic in \tilde{V}_1 . Finally, let $\tilde{V} = q_{\alpha_1}^{-1}(G)$. It is easily seen that \tilde{V} is an open extension of V and $\tilde{V} \subset W \subset \tilde{U}$. Consider a map $f : \mathbb{S}^k \rightarrow \tilde{V}$, where $k \leq n$. Then there exists a map $g : \mathbb{S}^k \rightarrow V_1$ such that $q_{\alpha_1} \circ f$ and g are homotopic in \tilde{V}_1 . We are going to show that g is homotopic to a constant map in the set U_1 . This will be done if the inclusion $j_1 : V_1 \hookrightarrow U_1$ generates a trivial homomorphism $j_1^* : \pi_k(V_1) \rightarrow \pi_k(U_1)$.

To this end, we consider the following commutative diagram:

$$\begin{array}{ccc} \pi_k(V_1) & \xrightarrow{j_1^*} & \pi_k(U_1) \\ \downarrow (p_{\alpha_0}^{\alpha_1})^* & & \downarrow (p_{\alpha_0}^{\alpha_1})^* \\ \pi_k(V_0) & \xrightarrow{j_0^*} & \pi_k(U_0) \end{array}$$

Since X_{α_1} is LC^n and the map $p_{\alpha_0}^{\alpha_1}$ is UV^n , each fiber of $p_{\alpha_0}^{\alpha_1}$ is an UV^n -set in X_{α_1} , see Proposition 2.5. Then, according to [8, Theorem 5.3], both vertical homomorphisms from the above diagram are isomorphisms. This implies that j_1^* is trivial because so is j_0^* . Hence, $q_{\alpha_1} \circ f$ is homotopic to a constant map in the set $\tilde{V}_1 \cup U_1 \subset q_{\alpha_1}(W)$ (recall that $q_{\alpha_1} \circ f$ is homotopic to g in \tilde{V}_1 and g is homotopic to a constant map in U_1). Therefore, there is a map $f_1: \mathbb{B}^{n+1} \rightarrow q_{\alpha_1}(W)$ extending $q_{\alpha_1} \circ f$. Since q_{α_1} is a soft map, and $W = q_{\alpha_1}^{-1}(q_{\alpha_1}(W))$, f can be extended to a map $\tilde{f}: \mathbb{B}^{n+1} \rightarrow W$. Thus, the inclusion $\tilde{V} \hookrightarrow \tilde{U}$ generates a trivial homomorphism between $\pi_k(\tilde{V})$ and $\pi_k(\tilde{U})$ for any $k \leq n$. So, X is ALC^n .

Now, suppose X is ALC^n , and consider X as a subset of some \mathbb{I}^B . Since the sets V and \tilde{V} in the ALC^n definition depend on the point x , the set U and its open extension \tilde{U} , respectively, we use the notations $\lambda(x, U) = V$ and $\lambda(x, U, \tilde{U}) = \tilde{V}$. First, we show that the sets V and \tilde{V} can be chosen to be functionally open in X and in \mathbb{I}^B , respectively. Indeed, if $U \subset X$ is a neighborhood of $x \in X$, we take a functionally open in X neighborhood V^* of x with $V^* \subset \lambda(x, U)$. Then for a given open in \mathbb{I}^B extension \tilde{U} of U and every $y \in V^*$ choose a functionally open in \mathbb{I}^B neighborhood $G(y)$ of y with $G(y) \subset \lambda(x, U, \tilde{U}) \cap G$, where G is an open in \mathbb{I}^B extension of V^* . Since V^* , as a functionally open subset of X , is Lindelöf, there exist countably many sets $G(y_i)$ whose union covers V^* . Obviously the set $\tilde{G} = \bigcup_{i=1}^{\infty} G(y_i)$ is a functionally open in \mathbb{I}^B extension of V^* which is contained in $\lambda(x, U, \tilde{U})$. So, every map from \mathbb{S}^k to \tilde{G} , $k \leq n$, is homotopic in \tilde{U} to a constant map.

Therefore, for every open set $U \subset X$ and every $x \in U$ there exists a functionally open in X set $V = \lambda(x, U)$ such that for any open in \mathbb{I}^B extension \tilde{U} of U we can find a functionally open in \mathbb{I}^B extension $\lambda(x, U, \tilde{U})$ of V contained in \tilde{U} with all homomorphisms $\pi_k(\lambda(x, U, \tilde{U})) \rightarrow \pi_k(\tilde{U})$, $k \leq n$, being trivial. If U is functionally open in X , then it is Lindelöf and there are countably many $x_i \in U$ such

that $\{\lambda(x_i, U) : i = 1, 2, \dots\}$ is a cover of U . We fix such a countable cover $\gamma(U)$ for any functionally open set $U \subset X$.

Let $A \subset B$ and W_0, W_1, \dots, W_k be elements of the standard open base \mathcal{B}_A for \mathbb{I}^A such that

$$(w) \quad \emptyset \neq \overline{W_0} \cap X_A \subset \bigcup_{i=1}^{i=k} W_i.$$

Here, $X_A = q_A(X)$, where $q_A : \mathbb{I}^B \rightarrow \mathbb{I}^A$ is the projection. We also denote by $[W_0, W_1, \dots, W_k]_A$ the set $q_A^{-1}((\bigcup_{i=1}^{i=k} W_i) \setminus (X_A \setminus W_0))$. Observe that $[W_0, W_1, \dots, W_k]_A \cap X = q_A^{-1}(W_0) \cap X$, so $[W_0, W_1, \dots, W_k]_A \cap X$ is functionally open in X . Moreover, if $\gamma(q_A^{-1}(W_0) \cap X) = \{V(y_i) : i = 1, 2, \dots\}$, where $y_i \in q_A^{-1}(W_0) \cap X$ for all i , we consider the sets $\widetilde{V(y_i)} = \lambda(y_i, q_A^{-1}(W_0) \cap X, [W_0, W_1, \dots, W_k]_A)$. So, each $\widetilde{V(y_i)}$ is a functionally open extension of $V(y_i)$ and all homomorphisms $\pi_k(\widetilde{V(y_i)}) \rightarrow \pi_k([W_0, W_1, \dots, W_k]_A)$, $k \leq n$, are trivial. Denote the family $\{\widetilde{V(y_i)} : i = 1, 2, \dots\}$ by $\widetilde{\gamma}([W_0, W_1, \dots, W_k]_A)$. We say that a set $A \subset B$ is admissible if the following holds:

- $q_A^{-1}(q_A(V)) = V$ for all $V \in \widetilde{\gamma}([W_0, W_1, \dots, W_k]_A)$ and all finitely many elements W_0, W_1, \dots, W_k of \mathcal{B}_A satisfying condition (w);
- $p_A^{-1}(p_A(V)) = V$ for all $V \in \gamma(q_A^{-1}(W) \cap X)$ and $W \in \mathcal{B}_A$, where $p_A : X \rightarrow X_A$ is the restriction $q_A|_X$.

Recall that for any functionally open set U in \mathbb{I}^B (resp., in X) there is a countable set $s(U) \subset B$ such that $q_{s(U)}^{-1}(q_{s(U)}(U)) = U$ (resp., $p_{s(U)}^{-1}(p_{s(U)}(U)) = U$).

Claim 2. For any set $A \subset B$ there exists an admissible set $C \subset B$ of cardinality $|A| \cdot \aleph_0$ containing A .

We construct by induction sets $A = A_0 \subset A_1 \subset \dots \subset A_k \subset A_{k+1} \subset \dots$ of cardinality $|A| \cdot \aleph_0$ such that:

- $s(V) \subset A_{k+1}$ for all $V \in \gamma(q_{A_k}^{-1}(W) \cap X)$ and all $W \in \mathcal{B}_{A_k}$.
- $s(V) \subset A_{k+1}$ for all $V \in \widetilde{\gamma}([W_0, W_1, \dots, W_m]_{A_k})$ and all finitely many $W_0, W_1, \dots, W_m \in \mathcal{B}_{A_k}$ satisfying condition (w).

The construction follows from the fact that the cardinality of each base \mathcal{B}_{A_k} is $|A| \cdot \aleph_0$ and the families $\gamma(q_{A_k}^{-1}(W) \cap X)$ and $\widetilde{\gamma}([W_0, W_1, \dots, W_m]_{A_k})$ are countable provided $W, W_0, \dots, W_m \in \mathcal{B}_{A_k}$. It is easily seen that the set $C = \bigcup_{k=1}^{\infty} A_k$ is as required.

Claim 3. X_A is an ALC^n -space for every admissible set $A \subset B$.

Let $y \in U$, where U is open in X_A . Take $x \in X$ and $W_0 \in \mathcal{B}_A$ containing y such that $\overline{W_0} \cap X_A \subset U$ and $q_A(x) = y$. Then x belongs to some $V_x \in \gamma(q_A^{-1}(W_0) \cap X)$. Since $s(V_x) \subset A$ (recall that A is admissible), $p_A^{-1}(p_A(V_x)) = V_x$. So, $V = p_A(V_x)$ is a functionally open

in X_A neighborhood of y , which is contained in U . Take any open extension $\tilde{U} \subset \mathbb{I}^A$ of U , and finitely many $W_1, \dots, W_k \in \mathcal{B}_A$ satisfying $\overline{W_0} \cap X_A \subset \bigcup_{i=1}^{i=k} W_i \subset \tilde{U}$. Since $V_x \in \gamma(q_A^{-1}(W_0) \cap X)$, the set $\tilde{V}_x = \lambda(x, q_A^{-1}(W_0) \cap X, [W_0, W_1, \dots, W_k]_A)$ is a functionally open extension of V_x with $\tilde{V}_x \in \tilde{\gamma}([W_0, W_1, \dots, W_k]_A)$. Then $s(\tilde{V}_x) \subset A$ and $\tilde{V} = q_A(\tilde{V}_x)$ is a functionally open in \mathbb{I}^A extension of V . We are going to show that all homomorphisms $\pi_k(\tilde{V}) \rightarrow \pi_k(\tilde{U})$, $k \leq n$, are trivial. Indeed, every map $f: \mathbb{S}^k \rightarrow \tilde{V}$ can be lifted to a map $g: \mathbb{S}^k \rightarrow \tilde{V}_x$ because $q_A^{-1}(\tilde{V}) = \tilde{V}_x$ and q_A is a soft map. Recall that \tilde{V}_x belongs to $\tilde{\gamma}([W_0, W_1, \dots, W_k]_A)$, so g can be extended to a map $\tilde{g}: \mathbb{B}^{k+1} \rightarrow [W_0, W_1, \dots, W_k]_A$. Finally, $q_A \circ \tilde{g}: \mathbb{B}^{k+1} \rightarrow q_A([W_0, W_1, \dots, W_k]_A) \subset \tilde{U}$ is an extension of f . This completes the proof of Claim 3.

Claim 4. Let $A_2 \subset A_1$ be admissible subsets of B . Then each fiber of $p_{A_2}^{A_1}: X_{A_1} \rightarrow X_{A_2}$ is UV^n .

Let $x \in X_{A_2}$ and $U \subset \mathbb{I}^{A_1}$ be an open set containing $F = (p_{A_2}^{A_1})^{-1}(x)$. Take $W'_0 \in \mathcal{B}_{A_2}$ with $x \in W'_0$ and $(p_{A_2}^{A_1})^{-1}(\overline{W'_0} \cap X_{A_2}) \subset U$. So, $(q_{A_2}^{A_1})^{-1}(\overline{W'_0}) \cap X_{A_1} \subset U$. Next, choose $W_1, \dots, W_k \in \mathcal{B}_{A_1}$ such that $(q_{A_2}^{A_1})^{-1}(\overline{W'_0}) \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Obviously, $W_0 = (q_{A_2}^{A_1})^{-1}(W'_0) \in \mathcal{B}_{A_1}$ and $\overline{W_0} \cap X_{A_1} \subset \bigcup_{i=1}^{i=k} W_i$. Let $y \in X$ with $q_{A_2}(y) = x$. Then y belongs to some $V_y \in \gamma(q_{A_1}^{-1}(W_0) \cap X)$. Because $q_{A_1}^{-1}(W_0) = q_{A_2}^{-1}(W'_0)$ and A_2 is admissible, we have $s(V_y) \subset A_2$. Hence, $p_{A_2}^{-1}(p_{A_2}(V_y)) = V_y$ and $p_{A_2}^{-1}(x) = p_{A_2}^{-1}(F) \subset V_y$. Then $\tilde{V}_y = \lambda(y, q_A^{-1}(W_0) \cap X, [W_0, W_1, \dots, W_k]_{A_1})$ is a functionally open extension of V_y and $\tilde{V}_y \in \tilde{\gamma}([W_0, W_1, \dots, W_k]_{A_1})$. Because $s(\tilde{V}_y) \subset A_1$, $q_{A_1}^{-1}(q_{A_1}(\tilde{V}_y)) = \tilde{V}_y$ and $V = q_{A_1}(\tilde{V}_y)$ is an open subset of \mathbb{I}^{A_1} such that $F \subset V \subset \bigcup_{i=1}^{i=k} W_i \subset U$. Then, as in the proof of Claim 3, we can show that the inclusion $V \hookrightarrow U$ generates trivial homomorphisms $\pi_k(V) \rightarrow \pi_k(U)$. Hence, F is UV^n .

Claim 5. The union of any increasing sequence of admissible subsets of B is also admissible.

This claim follows directly from the definition of admissible sets.

Now we can complete the proof of Theorem 3.1. According to Claim 2 and Claim 5, the set B is covered by a family \mathcal{S} of countable sets such that \mathcal{S} is stable with respect to countable unions. Then, by Claim 3, each X_A , $A \in \mathcal{S}$, is a metric ALC^n -compactum. Hence, Proposition 2.7 yields that all spaces X_A , $A \in \mathcal{S}$, are LC^n . Moreover, the projections $p_{A_2}^{A_1}$ are UV^n -maps for any $A_1, A_2 \in \mathcal{S}$ with $A_2 \subset A_1$. Because the set B is admissible, it follows from Claim 4 that the limit projections

$p_A: X \rightarrow X_A$, $A \in \mathcal{S}$, are also UV^n -maps. Therefore, X is limit space of the σ -complete inverse system $\{X_A, p_{A_1}^{A_1}, A, A_1, A_2 \in \mathcal{S}\}$. \square

Corollary 3.2. *Any LC^n -compactum X is an ALC^n -space.*

Proof. We embed X in some \mathbb{I}^B and let $A_0 \subset B$ be a countable set. According to the factorization theorem of Bogaty-Smirnov [2, Theorem 3], there is a metric compactum Y_1 and maps $g_1: X \rightarrow Y_1$, $h_1: Y_1 \rightarrow X_{A_0}$ such that $p_{A_0} = h_1 \circ g_1$ and all fibers of g_1 are UV^n -sets in X . Then, by [8, Theorem 5.4], Y_1 is LC^n . Since g_1 depends on countably many coordinates, there is a countable set $A_1 \subset B$ containing A_0 and a map $f_1: X_{A_1} \rightarrow Y_1$ such that $f_1 \circ p_{A_1} = g_1$. In this way we construct countable sets $A_k \subset A_{k+1} \subset B$ and LC^n metric compacta Y_k together with maps $g_k: X \rightarrow Y_k$, $f_k: X_{A_k} \rightarrow Y_k$ and $h_k: Y_k \rightarrow X_{A_{k-1}}$ such that $g_k = f_k \circ p_{A_k}$, $p_{A_{k-1}} = h_k \circ g_k$ and the fibers of each g_k are UV^n -sets in X . Let A be the union of all A_k . Then X_A is the limit space of the inverse sequence $\mathcal{S} = \{Y_k, s_k^{k+1} = f_k \circ h_{k+1}\}$. According to [8, Theorem 5.3], for all open sets $U \subset Y_k$ the group $\pi_m(U)$ is isomorphic to $\pi_m(g_k^{-1}(U))$, $m = 0, 1, \dots, n$. This property of the maps g_k implies that any $s_k^{k+1}: Y_{k+1} \rightarrow Y_k$ is an UV^n -map. Hence, by Theorem 3.1, X_A is an ALC^n -compactum (as the limit of an inverse sequence of metric LC^n -compacta and bounding UV^n -maps). Finally, by Theorem 2.7, X_A is also an LC^n -space. Moreover, for any $y \in X_A$ we have $p_A^{-1}(y) = \bigcap_{k \geq 1} g_k^{-1}(y_k)$, where $y_k = s_k(y)$ with $s_k: X_A \rightarrow Y_k$ being the projections of \mathcal{S} . Because all $g_k^{-1}(y_k)$ are UV^n -sets in X , so is the set $p_A^{-1}(y)$. Therefore, every countable subset A_0 of B is contained in an element of the family \mathcal{A} consisting of all countable sets $A \subset B$ such that X_A is LC^n and the fibers of the map p_A are UV^n -sets in X . It is easily seen that the union of an increasing sequence of elements of \mathcal{A} is again from \mathcal{A} , and that $p_A^C: X_C \rightarrow X_A$ is an UV^n -map for all $A, C \in \mathcal{A}$ with $A \subset C$. So, the inverse system $\{X_A, p_A^C, A, C \in \mathcal{A}\}$ is σ -complete and consists of metric LC^n -compacta and UV^n -bounding maps. Then by Theorem 3.1, X is ALC^n (observe that the proof of the "if" part of Theorem 3.1 does not need the assumption that all limit projections are UV^n -maps). \square

Theorem 3.1 shows that the class of ALC^n -compacta is adequate to the class of UV^n -maps. Next theorem provides another classes of compacta adequate to all continuous maps.

Theorem 3.3. *A compactum X is a cell-like (resp., UV^n) space if and only if X is the limit space of a σ -complete inverse system consisting of cell-like (resp., UV^n) metric compacta.*

Proof. Suppose X is a cell-like compactum. Because this means that X has a shape of a point, we can apply Corollary 8.4.8 from [3] stating that if φ is a shape isomorphism between the limit spaces of two σ -complete inverse systems $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ and $\{Y_\alpha, q_\alpha^\beta, \alpha \in A\}$ of metric compacta, then the set of those $\alpha \in A$ for which there exist shape isomorphisms $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ satisfying $Sh(q_\alpha) \circ \varphi = \varphi_\alpha \circ Sh(p_\alpha)$ is cofinal and closed in A . So, according to this corollary, X is the limit space of a σ -complete inverse system consisting of metric cell-like compacta. In case X is an UV^n -compactum, it has an n -shape of a point (this notion was introduced by Chigogidze in [4]), and the above arguments apply.

Suppose now that X is the limit space of a σ -complete inverse system $\{X_\alpha, p_\alpha^\beta, \alpha \in A\}$ such that all X_α are metric cell-like compacta. As in the proof of Theorem 3.1, we can embed X in a Tychonoff cube \mathbb{I}^B , where B is the union of countable sets B_α , $\alpha \in A$, such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$, $B_\gamma = \bigcup \{B_{\gamma(k)} : k = 1, 2, \dots\}$ for any chain $\gamma(1) < \gamma(2) < \dots$ with $\gamma = \sup\{\gamma(k) : k \geq 1\}$, and each $p_\alpha^\beta : X_\beta \rightarrow X_\alpha$ is the restriction of the projection $q_\alpha^\beta : \mathbb{I}^{B_\beta} \rightarrow \mathbb{I}^{B_\alpha}$. Then $X_\alpha = q_\alpha(X) \subset \mathbb{I}^{B_\alpha}$ with q_α being the projection from \mathbb{I}^B onto \mathbb{I}^{B_α} . If U is a neighborhood of X in \mathbb{I}^B , there is α and an open set U_α in \mathbb{I}^{B_α} such that $q_\alpha^{-1}(U_\alpha) \subset U$. Since X_α is a cell-like space, there exists a closed neighborhood $V_\alpha \subset \mathbb{I}^{B_\alpha}$ of X_α contractible in U_α . Using that q_α is a soft map, we conclude that $q_\alpha^{-1}(V_\alpha)$ is contractible in $q_\alpha^{-1}(U_\alpha)$. Similarly, we can show that any limit space of a σ -complete inverse system of metric UV^n -compacta is also an UV^n -compactum. \square

REFERENCES

- [1] P. Bacon, *Extending a complete metric*, Amer. Math. Monthly **75** (1968) 642–643.
- [2] S. Bogatyĭ and Ju. Smirnov, *Approximation by polyhedra and factorization theorems for ANR-bicomplexa*. Fund. Math. **87** (1975), no. 3, 195–205 (in Russian).
- [3] A. Chigogidze, *Inverse spectra*, North-Holland Math. Library **53** (Elsevier Sci. B.V., Amsterdam, 1996)
- [4] A. Chigogidze, *The theory of n -shapes*, Russian Math. Surveys **44** (1989), 145–174.
- [5] A. Dranishnikov, A private communication 1990.
- [6] A. Dranishnikov, *Universal Menger compacta and universal mappings*, Math. USSR Sb. **57** (1987), no. 1, 131–149.
- [7] A. Dranishnikov, *Absolute ekstensors in dimension n and n -soft maps increasing dimension*, Uspekhi Mat. Nauk **39** (1984), no. 5(239), 55–93 (in Russian).
- [8] J. Dugundji, *Modified Vietoris theorem for homotopy*, Fund. Math. **66** (1970), 223–235.

- [9] J. Dugundji and E. Michael, *On local and uniformly local topological properties*, Proc. Amer. Math. Soc. **7** (1956), 304–307.
- [10] V. Gutev, *Selections for quasi-l.s.c. mappings with uniformly equi- LC^n range*, Set-Valued Anal. **1** (1993), no. 4, 319–328.
- [11] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Illinois Math. Joourn. **4** (1960), no. 2, 278–291.
- [12] N. To Nhu, *Investigating the ANR-property of metric spaces*, Fund. Math. **124** (1984), 243–254; Corrections: Fund. Math. **141** (1992), 297.
- [13] S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, 1965.
- [14] A. Karashev and V. Valov, *Extension dimension and quasi-finite CW-complexes*, Topology Appl. **153** (2006), 3241–3254.
- [15] B. Pasynkov, *Monotonicity of dimension and open mappings that raise dimension*, Trudy Mat. Inst. Steklova **247** (2004), 202–213 (in Russian).
- [16] B. Pasynkov, *On universal bcompacta of given weight and dimension*, Dokl. Akad. Nauk SSSR **154** (1964), no. 5, 1042–1043 (in Russian).
- [17] K. Sakai, *Geometric aspects of general topology*, Springer Monographs in Mathematics. Springer, Tokyo, 2013.
- [18] E. Shchepin, *Topology of limit spaces with uncountable inverse spectra*, Uspekhi Mat. Nauk **31** (1976), no. 5(191), 191–226 (in Russian).

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